Noise due to Pulse-to-Pulse Incoherence in Injection-Locked Pulsed-Microwave Oscillators. Part II—Effects of Phase-Locking Dynamics

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***Abstract*—The problem of noise due to partial pulse-to-pulse coherence in phase-locked pulsed oscillators is investigated. In particular, the analysis includes the dynamic time variation of the phase-locking process. The signal-to-noise ratio of such a system is found to increase as $(\gamma\tau)^2$, where γ is the frequency locking bandwidth and τ is the pulse length. This result corrects a previous conjecture of an exponential dependence on $\gamma\tau$.**

I. Introduction

INJECTION LOCKING plays a doubly beneficial role for oscillator systems by providing a stable output frequency as well as by suppressing the inherent noise level of the oscillator [1].

In addition to these well-known properties for a CW oscillator system, injection locking also provides a stable initial phase for *pulsed* systems. This has an important noise-suppressing effect, since otherwise the randomness of the initial phases for the individual pulses would give rise to an excess noise, which could well prove the dominant noise process for the output signal [2], [3].

In a previous work [3], we analyzed the importance of pulse-to-pulse coherence for achieving high signal-to-noise ratios in pulsed-oscillator systems. The effect of the phase-locking process was modeled by assuming the initial phases of the individual pulses to have a random variation, normally distributed with a phase spread $\langle(\Delta\phi)^2\rangle^{1/2}$ around the mean phase $\langle\phi\rangle$.

However, this approach neglected the fact that phase locking is a dynamic process, which continuously, during pulses, tends to improve the pulse-to-pulse coherence by mapping an initial maximum phase spread of 2π on a phase interval $2\Delta\phi(t)$ which is shrinking in time towards zero. Thus, a more detailed analysis of noise due to partial pulse-to-pulse coherence should include the dynamics of the phase-locking process. The purpose of the present work is to provide such an analysis.

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II. Review of Previous Results

If $f(t)$ denotes the (complex) amplitude variation of a single unit pulse, the total amplitude $g(t)$ for a pulsed system consisting of $(2N+1)$ pulses can be written

$$g(t)=\sum_{k=-N}^{+N} f(t-kT)\exp(i\phi_k) \tag{1}$$

where T is the pulse repetition time and ϕ_k denotes the phase of the kth pulse. In our previous study, we assumed $\{\phi_k\}$ to constitute a normal random process with an rms phase spread $(\langle(\Delta\phi)^2\rangle)^{1/2}$. It was then shown that the normalized power spectrum $G_0(\omega)$ could be obtained by Fourier analyzing (1) together with a subsequent statistical averaging. This yielded

$$G_0(\omega)\equiv\frac{1}{(2N+1)}\left\langle|G(\omega)|^2\right\rangle = [\mu S_0(\omega)+(1-\mu)]|F(\omega)|^2 \tag{2}$$

where $G(\omega)$ and $F(\omega)$ are the transforms of $g(t)$ and $f(t)$, respectively, and $S_0(\omega)$ is the coherent sampling function

$$S_0(\omega)=\frac{1}{(2N+1)}\frac{\sin^2[(N+1/2)\omega T]}{\sin^2\frac{\omega T}{2}}. \tag{3}$$

The weighting factor μ is determined by the phase spread as

$$\mu=\exp\left(-\left\langle(\Delta\phi)^2\right\rangle\right). \tag{4}$$

We emphasized that (2) provides a suggestive description of the influence of partial pulse-to-pulse coherence on the power spectrum by being a weighted mean of a completely coherent part $(\mu S_0(\omega))$ and a completely incoherent part $(1-\mu)$.

III. Phase-Locking Dynamics

However, the phases of the individual pulses actually evolve in time according to the dynamic phase-locking equation [1]

$$\frac{d\phi_k(t)}{dt}=\Delta\omega_0-\gamma\sin\phi_k(t) \tag{5}$$

where $\Delta\omega_0$ is the difference between the frequencies of the locking signal and the free-running oscillator, and γ is the maximum frequency offset for which locking can be achieved. γ is determined by the parameters of the oscillator together with the ratio of the amplitudes of the free-running oscillator and the injected signal [1].

The characteristic locking phase ϕ_L is obtained from (5) as

$$\sin\phi_L = -\frac{\Delta\omega_0}{\gamma}.$$

For simplicity, we will concentrate on the case of exact resonance ($\Delta\omega_0 = 0$), when the stable locking phase becomes $\phi_L = 0$. The phase variation during the locking process is obtained by solving (5), assuming an initial phase ϕ_k. The solution becomes particularly simple for small ϕ_k, viz.,

$$\phi_k(t) = \phi_k \exp(-\gamma t). \tag{6}$$

Equation (6) yields the dynamic time evolution of the phases of the individual pulses during the locking process towards the common phase $\phi_L = 0$.

In the Appendix, we briefly discuss the consequences of allowing for nonresonant locking processes and also the quality of the exponential approximation of the phase-locking variation (6).

IV. Power Spectrum in the Presence of Dynamic Phase Locking

When the phases of the individual pulses vary according to (6), we can write the signal as, cf. (1)

$$g(t) = \sum_{k=-N}^{+N} f(t-kT)h_k(t-kT) \tag{7}$$

where

$$h_k(t) = \exp\left[i\phi_k \exp(-\gamma t)\right].$$

By taking the Fourier transform of (7), assuming a rectangular unit pulse of length τ, and expanding $h_k(t)$ as a power series in the variable $i\phi_k\exp(-\gamma t)$, we obtain the (unaveraged) power spectrum as

$$|G(\omega)|^2 = \sum_{k,1=-N}^{+N} e^{-i(k-1)\omega T} \cdot \sum_{m,n=0}^{\infty} \frac{(i\phi_k)^n(-i\phi_1)^m}{n!m!} g_{n,m}(\omega)|F(\omega)|^2 \tag{8}$$

where we have introduced the notation

$$g_{n,m}(\omega) = g_n(\omega)g_m^*(\omega) \tag{9}$$

with

$$g_n(\omega) = \frac{1-\exp\left[-(i\omega+n\gamma)\tau\right]}{1-\exp(-i\omega\tau)} \frac{1}{1-in\gamma/\omega} \tag{10}$$

and $F(\omega)$ is the spectrum of the rectangular unit pulse. In order to proceed to the statistical averaging of $|G(\omega)|^2$, we note that the initial phases ϕ_k must be assumed to be randomly distributed over the interval $[-\pi, +\pi]$ with a constant probability function $p(\phi_k) = 1/(2\pi)$. Furthermore ϕ_k and ϕ_1 are uncorrelated if $k \neq 1$. Thus

$$\langle\phi_k^n\phi_1^m\rangle = \begin{cases}\langle\phi_k^n\rangle\langle\phi_1^m\rangle, & \text{if } k\neq 1\\ \langle\phi_k^{n+m}\rangle, & \text{if } k=1\end{cases} \tag{11}$$

and

$$\langle\phi_k^n\rangle = \begin{cases}0, & \text{if } n \text{ is odd}\\ \dfrac{\pi^n}{n+1}, & \text{if } n \text{ is even.}\end{cases} \tag{12}$$

The averaged power spectrum can again be suggestively presented as a sum of a coherent and an incoherent part, cf. (2). We find from (8) using (11) and (12)

$$G_0(\omega) = \frac{1}{(2N+1)}\left\langle|G(\omega)|^2\right\rangle = \left[h_1(\omega)S_0(\omega)+h_2(\omega)-h_1(\omega)\right]|F(\omega)|^2 \tag{13}$$

where

$$h_1(\omega) = \left|\sum_{k=0}^{\infty}(-1)^k\frac{\pi^{2k}}{(2k+1)!}g_{2k}(\omega)\right|^2 \tag{14}$$

$$h_2(\omega) = \sum_{k=0}^{\infty}(-1)^k\frac{\pi^{2k}}{(2k+1)!} \cdot \sum_{m=0}^{2k}(-1)^m\binom{2k}{m}g_{2k-m,m}(\omega). \tag{15}$$

Equations (13)–(15) constitute the proper generalization of our previous results to include dynamic phase locking.

V. Signal-to-Noise Ratio for Strongly Phase-Locked Pulsed Systems

Although the general result for the power spectrum (13)–(15) is in a physically suggestive form, it is not very explicit, in view of the complicated expressions for $h_1(\omega)$ and $h_2(\omega)$.

However, in two special limits, the power spectrum degenerates into well-known forms. This fact also constitutes a check on the results.

i) In the limit of $\gamma\tau \to 0$, the phase-locking mechanism is not operating and we should regain the completely incoherent result.

When $\gamma\tau \to 0$, we obtain $g_n(\omega) \to 1$ for all n and

$$h_1(\omega) \to \left(\sum_{k=0}^{\infty}(-1)^k\frac{\pi^{2k}}{(2k+1)!}\right)^2 = \left(\frac{\sin\pi}{\pi}\right)^2 = 0 \tag{16}$$

which implies that the coherent part vanishes, cf. (13). For $h_2(\omega)$, we find

$$h_2(\omega) \to \sum_{k=0}^{\infty}(-1)^k\frac{2k}{(2k+1)!}\sum_{m=0}^{2k}(-1)^m\binom{2k}{m} = 1 \tag{17}$$

since the inner sum is zero, except for $k = 0$ when it becomes equal to one. Thus, in the limit $\gamma\tau \to 0$, $G_0(\omega) = |F(\omega)|^2$, i.e., the completely incoherent result, as expected.

ii) In the limit $\gamma\tau \to \infty$, phase locking is instantaneous and we should regain the completely coherent result.

When $\gamma\tau \to \infty$, we obtain $g_n(\omega) = 0$ for $n \neq 0$ and $g_0(\omega) = 1$, which implies that $h_1(\omega) \to 1$ and $h_2(\omega) \to 1$. Thus, the incoherent part of $G_0(\omega)$ vanishes and we regain the completely coherent result $G_0(\omega) = S_0(\omega)|F(\omega)|^2$, as we should.

Most technically important situations involving pulsed phase-locked oscillators can be considered as *strongly* phase locked in the sense that $\gamma\tau \gg 1$, i.e., the characteristic locking time, $1/\gamma$, cf. (6), is much less than the pulse duration time τ. Typical values for certain modern pulsed-radar transmitter systems could be $\gamma \simeq 100$ MHz and $\tau \simeq 800$ ns, implying that $\gamma\tau \simeq 5 \times 10^2$.

In the case $\gamma\tau \gg 1$, $g_n(\omega)$ simplifies to $(g_0(\omega) = 1)$

$$g_n(\omega) \simeq i\frac{\omega}{n\gamma}\frac{1}{1-\exp(-i\omega T)}, \qquad n \neq 0 \tag{18}$$

and we find that

$$h_1(\omega) \simeq 1 - \frac{\omega c_0}{\gamma}\cot\frac{\omega T}{2} + \left(\frac{\omega}{2\gamma}\right)^2 \frac{c_0^2}{\sin^2(\omega\tau/2)} \tag{19}$$

$$h_2(\omega) \simeq 1 - \frac{\omega c_0}{\gamma}\cot\frac{\omega T}{2} + \left(\frac{\omega}{2\gamma}\right)^2 \frac{c_1}{\sin^2(\omega\tau/2)} \tag{20}$$

where the constants c_0 and c_1 are defined by

$$c_0 = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\pi^{2k}}{(2k)(2k+1)!}$$

$$c_1 = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\pi^{2k}}{(2k+1)} a_{2k} \tag{21}$$

with

$$a_{2k} = \frac{1}{4k}\left(\sum_{m=1}^{2k} \frac{1}{m} + \frac{1}{2k}\right). \tag{22}$$

The series defining c_0 can be rewritten to yield

$$c_0 = \gamma_0 + \ln\pi - 1 - Ci(\pi) \simeq 0.65 \tag{23}$$

where γ_0 is Euler's constant and $Ci(x)$ denotes the cosine integral [4]. The series defining c_1 converges rapidly and we find

$$c_1 \simeq 0.60. \tag{24}$$

Equations (19) and (20) imply that the incoherent contribution to the power spectrum, in this limit, degenerates into white noise with a level determined by

$$[h_1(\omega) - h_2(\omega)]|F(\omega)|^2 \simeq \frac{c_1 - c_0^2}{\gamma^2 T^2} \simeq \frac{0.2}{\gamma^2 T^2}. \tag{25}$$

In particular, close to the main peak of the limit pulse spectrum ($\omega\tau \ll 1$), the signal-to-noise ratio S becomes

$$S \simeq 10 N \gamma^2 \tau^2. \tag{26}$$

This shows that the conjecture in [3] of an exponential increase with $\gamma\tau$ was too optimistic. Taking, as in [3], $2N \simeq 6 \times 10^4$ and assuming $\gamma\tau \simeq 5 \times 10^2$, we obtain $S \simeq 92$ dB.

Thus, our results demonstrate that partial pulse-to-pulse coherence could be an important noise source, which may well limit the performance of low-noise pulsed-oscillator systems.

However, in this context it is appropriate to point out that a major difference can be expected for the noise properties of pulsed phase-locked oscillator systems, depending on whether the locking signal is turned on before or after the rising edge of the oscillator pulse. The results presented in this paper are applicable to the case when the locking signal is introduced when the oscillator signal has reached its maximum flat top value. On the other hand, if the locking signal is introduced before the rising edge of the oscillator pulse, the effective locking bandwidth (being proportional to the ratio of the amplitudes of the locking and oscillator signals, respectively) is very large. This implies that the phase locking becomes much more efficient and that the noise level due to partial pulse-to-pulse coherence is correspondingly reduced as compared to the first case.

Appendix

In this Appendix, we will discuss two interesting points raised by the reviewers.

i) The present analysis assumes that the phase-locking dynamics can be described by an exponential decay towards the locking phase value, cf. (5). This simple solution is valid, provided that the initial phase does not deviate too much from the final locking phase. Actually, this contradicts the fact that we consider initial phases in the whole interval $[-\pi, \pi]$.

ii) The analysis only considers the case of exact resonance $\Delta\omega_0 = 0$, whereas most realistic situations are characterized by $\Delta\omega_0 \neq 0$.

In order to discuss the consequences of relaxing these limiting assumptions, we rewrite the phase-locking equation (5) as

$$\frac{d\phi}{d\tau} = K - \sin\phi \tag{A1}$$

where $\tau = \gamma t$ and $K = \Delta\omega_0/\gamma$.

We first consider the following case.

Exact Resonance with Finite Initial Phases

Exact resonance requires $K = 0$, in which case the full solution of (A1), subject to the initial condition $\phi(0) = \phi_0$, is

$$\phi = 2\arctan\left[\left(\tan\frac{\phi_0}{2}\right)e^{-\tau}\right] \tag{A2}$$

which should be compared to the approximate solution in the limit of small ϕ_0, i.e.,

$$\phi = \phi_0 \exp(-\tau). \tag{A3}$$

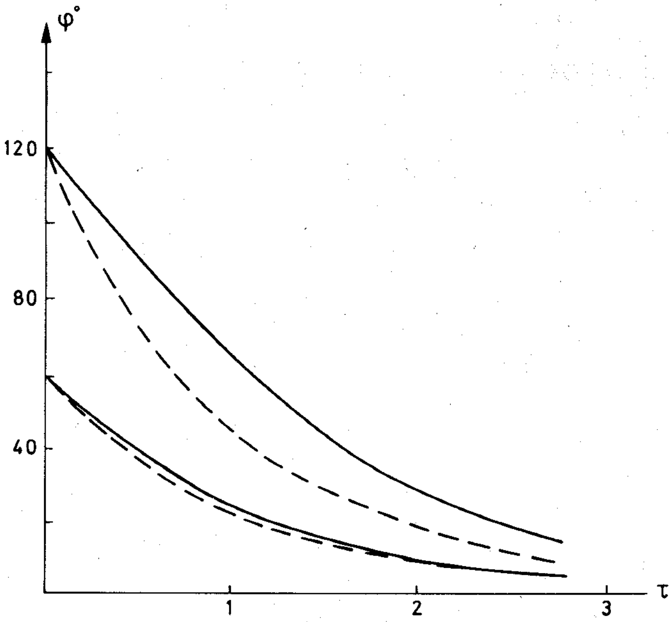

Fig. 1. Phase-locking curves for large initial phases and $\Delta\phi_0 = 0$, exact solution (———), (A2), and approximate solution (- - - -), (A3).

In Fig. 1, we compare the exact and approximate solutions for two different and comparatively large initial phases. The agreement is still very good at $\phi_0 = 60°$, has deteriorated somewhat at $\phi_0 = 120°$, and gets increasingly bad as we approach $\phi_0 = 180°$, the unstable stationary value of (A1) for $K = 0$. On the other hand, the time to lock the phase to within, e.g., 10° of the final value does not depend very strongly on the initial phase. Thus, our use of an exponential phase-locking variation, although not strictly valid for large initial phases, should be an acceptable approximation, but should admittedly result in a somewhat too optimistic estimate of the noise level.

Nonresonant Phase Locking

For nonresonant phase locking ($0 < |K| < 1$), the solution of (A1) is given by [5]

$$\phi - \phi_L = 2\left\{\arctan\frac{1}{K} - \frac{\sqrt{1-K^2}}{K}\tanh\left(\frac{\tau-\tau_0}{2}\sqrt{1-K^2}\right) - \arctan\left(\frac{1}{K} - \frac{\sqrt{1-K^2}}{K}\right)\right\} \quad \text{(A3)}$$

where ϕ_L denotes the asymptotic locking phase, viz.,

$$\phi_L = 2\arctan\left(\frac{1}{K} - \frac{\sqrt{1-K^2}}{K}\right) \quad \text{(A4)}$$

and τ_0 is an integration constant which is implicitly determined by the initial phase ϕ_0 as follows:

$$\phi_0 = 2\arctan\left[\frac{1}{K} + \frac{\sqrt{1-K^2}}{K}\tanh\left(\frac{\tau_0}{2}\sqrt{1-K^2}\right)\right]. \quad \text{(A5)}$$

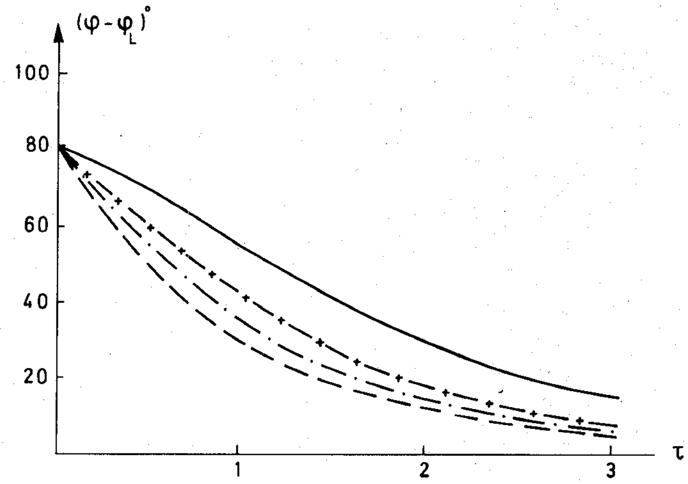

Fig. 2. Nonresonant phase-locking curves $K = 1/2$ (———), $K = 1/4$ (-x-x-) as compared to resonant phase locking $K = 0$, exact solution (-·-·-) and approximate (- - - -).

In Fig. 2, we compare phase locking as given by (A5) for different K with the previous results for exact resonance ($K = 0$) and the approximate exponential variation. It is clearly seen that $K = 0$ is not a necessary requirement for the applicability of the present analysis, e.g., for $K = 1/4$ the phase-locking curve does not significantly deviate from that of $K = 0$. Again, for large K and/or large initial phase differences ($\phi_0 - \phi_L$), the agreement deteriorates but we can conclude that, well within the locking bandwidth, the results for the noise levels should still be approximately valid.

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Accurate Analysis Equations and Synthesis Technique for Unilateral Finlines

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Abstract—**Accurate analysis equations and synthesis techniques are presented for unilateral finlines, valid over a wide range of structural parameters and substrate dielectric constants ($1 \leqslant \epsilon_r \leqslant 3.75$). These expressions are usable for computing the cutoff wavelength to within ± 0.6 percent, the guided wavelength to within ± 2 percent, and the characteristic impedance (based on the power–voltage definition) to within ± 2 percent, of the spectral-domain method, over the normalized frequency range** $0.25 \leqslant b/\lambda \leqslant 0.6$.

I. Introduction

FINLINE IS AN ideal transmission line for millimeter-wave circuits because it avoids miniaturization and offers the potential for low-cost production through batch processing techniques [1], [2]. It is also easily compatible with semiconductor devices. It has wide bandwidth for single-mode operation, moderate attenuation, and low dispersion in the frequency range of interest. These properties have made it more popular than microstrip about 30 GHz.

Dispersion in finline has been accurately analyzed by Hofmann [3], Knorr and Shayda [4], Schmidt and Itoh [5], Beyer and Wolff [6], Sharma, Costache, and Hoefer [7], Shih and Hoefer [8], and Saad and Schunemann [9]. These analyses use the eigenmode analysis in space or the spectral-domain, finite-element method, or a two-dimensional transmission-line matrix. The network analytical method of electromagnetic fields, which is similar to the spectral-domain technique, was extended to the more general case of higher order modes by Hayashi, Farr, and Mittra [10]. Although the above-mentioned methods are highly accurate, they require considerable analytical effort and lead to complicated computer programming.

Besides the rigorous analyses above, the propagation constant in finlines has been approximated by various methods. Many authors have treated finlines as ridged waveguides [11], [12]. But the resulting expressions are of poor accuracy for the guided wavelength and the characteristic impedance. For an adequately accurate expression for the effective dielectric constant of finlines, one has to depend on experimental data [1] from expensive and time-consuming sample measurements. Therefore, in spite of all the advantages of a novel transmission line, the basic problem faced by the designers is the cumbersome design procedure.

Consequently, there remains a strong need for accurate closed-form expressions for the equivalent dielectric constant and characteristic impedance for finlines. Recently, Sharma and Hoefer [13] have presented purely empirical expressions for the cutoff wavelength of unilateral and bilateral finlines, which were developed by curve fitting to numerical results obtained by the spectral-domain technique [7]. Because of their purely empirical nature, these expressions are valid for a small range of finline geometries. For example, the equations are valid for $1/16 \leqslant d/b \leqslant 1/4$, $b/a = 0.5$, and $\epsilon_r = 2.22$ and 3.00 only (see Fig. 1(a)). Moreover, different equations are required for dielectric substrates of different permittivity values.

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